A Novel Optimisation Scheme for Designing High Frequency Transformer Windings

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ABSTRACT

With the miniaturisation of magnetic components in power supplies, increased switching frequencies in the kHz-MHz range are required. But with these increased frequencies comes the problem of increased winding losses due to proximity effects. This paper describes how waveforms encountered in switch-mode power supplies may be incorporated into a transformer design algorithm, with emphasis on minimising ac resistances (and hence power dissipation) in the windings. Approximation formulæ for the optimum thickness of a foil have been found using regression analysis and Taylor series approximations for duty-cycle varying waveforms, and are given in terms of N, the number of harmonics, and p, the number of layers of foil required. These formulæ will be implemented as part of the winding selection process in a Windows-based package.

1. Waveform Analysis

The formula for the optimum thickness of a layer in a transformer winding may be derived for waveforms with varying and non-varying duty-cycles. This has been done for a number of waveforms as shown in Table 1, and the formula for a duty-cycle varying pulsed (or rectified square) waveform is now derived as a sample case.

The waveform shown in Figure 1 is representative of the current in a push-pull winding. I_o is related to the dc output current; for a 1:1 turns ratio, it is equal to the dc output current for a 100% duty cycle.

Figure 1 can be taken as an even function about 0 as shown in Figure 2. We shall take one period T (marked by dashed arrow) to calculate the Fourier Series of i.

For a range (-l, l), an even function has a Fourier Series of the type f(x):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \mathrm{Cos}\frac{n\pi x}{l} \qquad (1)$$

In our case, $l = T/2$, $x = t$ and $f(x) = i(t)$. Also, since $\omega = 2\pi/T$, $n\pi x/l = n\pi t2/T = n\omega t$.

$$i(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \mathrm{Cos}(n\omega t) \qquad (2)$$

Calculating the Fourier coefficients a_n and a_0 yields

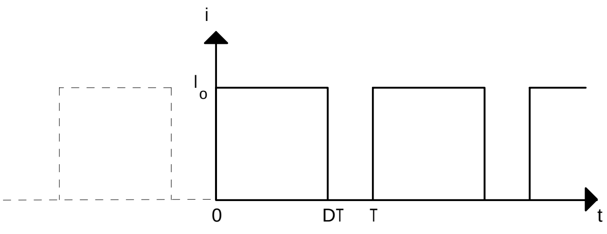

Figure 1: Pulsed current waveform with a duty-cycle of D

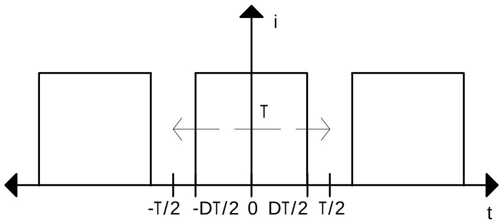

Figure 2: Same waveform taken as an even function

$$\begin{aligned} a_n &= \frac{2}{l}\int_0^l f(x)\mathrm{Cos}\frac{n\pi x}{l}dx = \frac{4}{T}\int_0^{\frac{T}{2}} i(t)\mathrm{Cos}(n\omega t)dt \\ &= \frac{4}{T}\left[\int_0^{\frac{DT}{2}} I_o \mathrm{Cos}(n\omega t)dt + \int_{\frac{DT}{2}}^{\frac{T}{2}} 0\mathrm{Cos}(n\omega t)dt\right] \\ &= \frac{2I_o}{n\pi}\mathrm{Sin}(n\pi D) \end{aligned} \qquad (3)$$

$$\begin{aligned} a_0 &= \frac{2}{l}\int_0^l f(x)dx = \frac{4}{T}\int_0^{\frac{T}{2}} i(t)dt \\ &= \frac{4}{T}\left[\int_0^{\frac{DT}{2}} I_o dt + \int_{\frac{DT}{2}}^{\frac{T}{2}} 0dt\right] = 2I_oD \end{aligned} \qquad (4)$$

These coefficients are then inserted into the expression for i(t) giving

$$i(t) = I_oD + \sum_{n=1}^{\infty}\frac{2I_o}{n\pi}\mathrm{Sin}(n\pi D)\mathrm{Cos}(n\omega t) \qquad (5)$$

The average value of current is $I_{dc} = I_oD$. The RMS value of current is $I_o\sqrt{D}$. Also, the Fourier Series of i can be expanded as

$$\begin{aligned} i(t) &= I_oD + \frac{2I_o}{\pi}\left(\frac{1}{1}\mathrm{Sin}(\pi D)\mathrm{Cos}(\omega t)\right. \\ &\left. + \frac{1}{2}\mathrm{Sin}(2\pi D)\mathrm{Cos}(2\omega t) + \frac{1}{3}\mathrm{Sin}(3\pi D)\mathrm{Cos}(3\omega t) + \ldots\right) \end{aligned} \qquad (6)$$

If D = 0.5, this reduces to

$$\begin{aligned} i(t) &= \frac{I_o}{2} + \frac{2I_o}{\pi}\left(\frac{1}{1}\mathrm{Cos}(\omega t) - \frac{1}{3}\mathrm{Cos}(3\omega t)\right. \\ &\left. + \frac{1}{5}\mathrm{Cos}(5\omega t) - \ldots\right) \end{aligned} \qquad (7)$$

The RMS value of the nth harmonic is

$$I_n = \frac{1}{\sqrt{2}}\left[\frac{2I_o}{n\pi}\mathrm{Sin}(n\pi D)\right] = \frac{\sqrt{2}I_o}{n\pi}\mathrm{Sin}(n\pi D) \qquad (8)$$

The total power loss is $P = R_{eff}I_{rms}^2$ which is made up of the dc component and the harmonics:

$$\begin{aligned} P &= R_{dc}I_{dc}^2 + R_{ac_1}I_1^2 + R_{ac_3}I_2^2 + \ldots \\ &= R_{dc}I_{dc}^2 + \sum_{n=1}^{\infty} R_{ac_n}I_n^2 \end{aligned} \tag{9}$$

R_{ac_n} is the ac resistance due to the nth harmonic, and is given by

$$R_{ac_n} = k_{p_n}R_{dc} \tag{10}$$

where k_{p_n} is the proximity effect factor due to the nth harmonic [1]. Thus, P is equal to

$$P = R_{dc}I_{dc}^2 + R_{dc}\sum_{n=1}^{\infty} k_{p_n}I_n^2 \tag{11}$$

Since $P = R_{eff}I_{rms}^2$, the above can be rearranged to give

$$\begin{aligned} \frac{R_{eff}}{R_{dc}} &= \frac{I_{dc}^2 + \sum_{n=1}^{\infty} k_{p_n}I_n^2}{I_{rms}^2} \\ &= \frac{I_o^2D^2 + \frac{2I_o^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\mathrm{Sin}^2(n\pi D)k_{p_n}}{I_o^2D} \\ &= D + \frac{2}{\pi^2 D}\sum_{n=1}^{\infty}\frac{1}{n^2}\mathrm{Sin}^2(n\pi D)k_{p_n} \end{aligned} \tag{12}$$

The pulse in Figure 1 is an ideal case. Normally, there would be a rise time and fall time associated with the waveform so that a finite number of harmonics are required. Typically, the upper limit on the number of harmonics is

$$N = \frac{35}{t_r\%} \tag{13}$$

where t_r is the percentage rise time and N is odd. For example, a 2.5% rise time would give N = 13.

Define R_δ as the dc resistance of a foil of thickness δ_o, where δ_o is the skin depth at the fundamental frequency of the pulsed waveform. R_{dc} is the dc resistance of a foil of thickness d and

$$\begin{aligned} &\frac{R_\delta}{R_{dc}} = \frac{d}{\delta_o} = \Delta \\ \Rightarrow\ &\frac{R_{eff}}{R_\delta} = \frac{R_{eff}/R_{dc}}{\Delta} \end{aligned} \tag{14}$$

The ratio R_{eff}/R_δ is given the name k_r, and for a given frequency, R_δ and δ_o are constant. Evidently, a plot of k_r versus Δ has the same shape as a plot of R_{eff} versus d.

The x-axis is increasing foil thickness. For $\Delta < \Delta_{opt}$, the dc resistance decreases as the thickness increases; however for $\Delta > \Delta_{opt}$, the ac effects are greater than the effect of increased thickness. The minimum ac resistance is given at Δ_{opt} and the optimum thickness is

$$d_{opt} = \Delta_{opt}.\delta_o \tag{15}$$

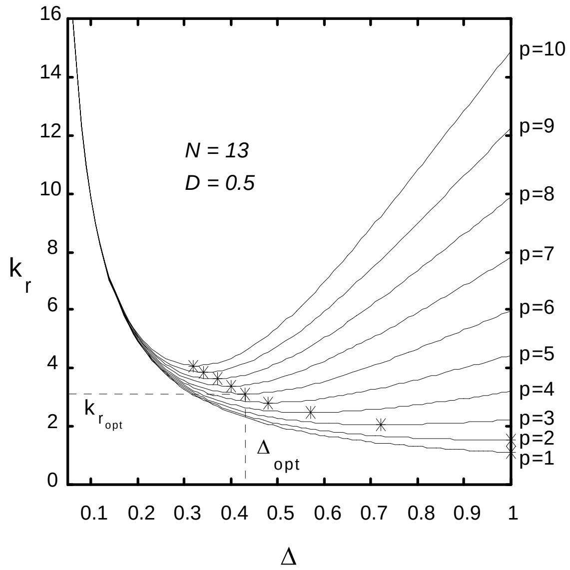

Figure 3: Plot of k_r versus Δ for 13 harmonics and various numbers of layers (D = 0.5)

The effective ac resistance of a foil of thickness d is

$$R_{eff} = k_rR_\delta = k_r\Delta R_{dc} \tag{16}$$

Assuming a maximum of N harmonics, k_r is obtained from (12) and (14):

$$\begin{aligned} k_r &= \frac{R_{eff}}{\Delta R_{dc}} \\ &= \frac{D}{\Delta} + \frac{2}{\pi^2 D}\sum_{n=1}^{N}\frac{\mathrm{Sin}^2(n\pi D)}{n^2\Delta}k_{p_n} \end{aligned} \tag{17}$$

k_{p_n} is given by Dowell's formula [1]:

$$k_{p_n} = \sqrt{n}\Delta\left[\frac{\mathrm{Sinh}(2\sqrt{n}\Delta)+\mathrm{Sin}(2\sqrt{n}\Delta)}{\mathrm{Cosh}(2\sqrt{n}\Delta)-\mathrm{Cos}(2\sqrt{n}\Delta)} + \frac{2(p^2-1)}{3}\frac{\mathrm{Sinh}(\sqrt{n}\Delta)-\mathrm{Sin}(\sqrt{n}\Delta)}{\mathrm{Cosh}(\sqrt{n}\Delta)+\mathrm{Cos}(\sqrt{n}\Delta)}\right] \tag{18}$$

where p is the number of layers of foil. The skin depth at the nth harmonic is

$$\begin{aligned} \delta_n &= \frac{1}{\sqrt{\pi n f \mu_r \mu_o \sigma}} = \frac{\delta_o}{\sqrt{n}} \\ \Delta_n &= \frac{d}{\delta_n} = \sqrt{n}\frac{d}{\delta_o} = \sqrt{n}\Delta \end{aligned} \tag{19}$$

k_r is now given by

$$\begin{aligned} k_r = \frac{D}{\Delta} + \frac{2}{\pi^2 D}\sum_{n=1}^{N}\frac{\mathrm{Sin}^2(n\pi D)}{n^{\frac{3}{2}}} \times \\ \left[\frac{\mathrm{Sinh}(2\sqrt{n}\Delta)+\mathrm{Sin}(2\sqrt{n}\Delta)}{\mathrm{Cosh}(2\sqrt{n}\Delta)-\mathrm{Cos}(2\sqrt{n}\Delta)} + \frac{2(p^2-1)}{3}\frac{\mathrm{Sinh}(\sqrt{n}\Delta)-\mathrm{Sin}(\sqrt{n}\Delta)}{\mathrm{Cosh}(\sqrt{n}\Delta)+\mathrm{Cos}(\sqrt{n}\Delta)}\right] \end{aligned} \tag{20}$$

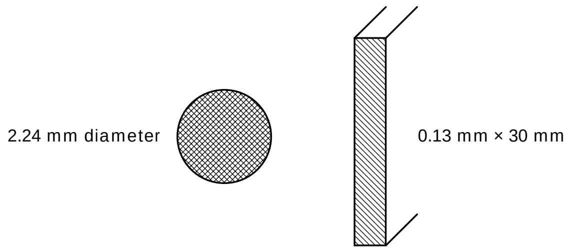

Figure 4: Round versus foil conductor

Example: Push-Pull Converter

Take D = 0.5, p = 6, t_r = 2.5%, and f = 50 kHz.

$$N = \frac{35}{t_r\%} = \frac{35}{2.5} = 14$$

Choose N = 13, since N is odd. From the graph of k_r versus Δ for p = 6 in Figure 3

$$k_{r_{opt}} = 3.12 \text{ and } \Delta_{opt} = 0.43$$

$$\delta_o = \frac{66}{\sqrt{f}} = \frac{66}{\sqrt{50 \times 10^3}} = 0.295 \text{ mm}$$

The optimum foil thickness is equal to

$$d_{opt} = \Delta_{opt}\delta_o = 0.43 \times 0.295 = 0.13 \text{ mm}$$

and the ac resistance is

$$R_{eff} = k_r \Delta R_{dc} = 3.12 \times 0.43 R_{dc} = 1.34 R_{dc}$$

A 2.24 mm diameter of bare copper wire has the same copper area as a 0.13 mm × 30 mm foil. The skin effect factor of the round conductor is given by

$$k_s = 0.25 + 0.5\left(\frac{r_o}{\delta_o}\right) = 0.25 + 0.5(3.8) = 2.15$$

This is for the fundamental frequency only. Evidently in this case, the choice of a foil conductor is vastly superior.

2. Approximate Analysis

The following general approximations can be made:

$$\frac{\text{Sinh}(2\Delta) + \text{Sin}(2\Delta)}{\text{Cosh}(2\Delta) - \text{Cos}(2\Delta)} \approx \frac{1}{\Delta} + \frac{\Delta^3}{a}$$
$$\frac{\text{Sinh}(\Delta) - \text{Sin}(\Delta)}{\text{Cosh}(\Delta) + \text{Cos}(\Delta)} \approx \frac{\Delta^3}{b} \qquad (21)$$

where a and b are constants. The values of a and b may be arrived at by using either of two methods. The first one involves expanding the trigonometric functions in (21) using Taylor's series and limiting them to a set number of terms:

$$\text{Cos}(\Delta) \approx 1 - \frac{\Delta^2}{2!} + \frac{\Delta^4}{4!} \quad \text{Cosh}(\Delta) \approx 1 + \frac{\Delta^2}{2!} + \frac{\Delta^4}{4!}$$
$$\text{Sin}(\Delta) \approx \Delta - \frac{\Delta^3}{3!} + \frac{\Delta^5}{5!} \quad \text{Sinh}(\Delta) \approx \Delta + \frac{\Delta^3}{3!} + \frac{\Delta^5}{5!} \qquad (22)$$

This method yields a = 7.5 and b = 6. Alternatively, a and b may be obtained by approximating the full trigonometric expressions in (21) using regression analysis over a particular range of Δ. This method yields a better approximate fit over that range. For Δ between 0.1 and 1.0, a = 11.57 and b = 6.18.

The proximity effect factor is then given by

$$k_p = \Delta\left[\frac{\text{Sinh}(2\Delta) + \text{Sin}(2\Delta)}{\text{Cosh}(2\Delta) - \text{Cos}(2\Delta)} + \frac{2(p^2-1)}{3}\frac{\text{Sinh}(\Delta) - \text{Sin}(\Delta)}{\text{Cosh}(\Delta) + \text{Cos}(\Delta)}\right]$$
$$\approx 1 + \frac{\Delta^4}{a} + \frac{2(p^2-1)}{3}\frac{\Delta^4}{b} \qquad (23)$$
$$= 1 + \left[\frac{2}{3b}p^2 + \frac{1}{a} - \frac{2}{3b}\right]\Delta^4$$

Substituting this expression into k_r gives

$$k_r = \frac{D}{\Delta} + \frac{2}{\pi^2 D}\sum_{n=1}^{N}\frac{\text{Sin}^2(n\pi D)}{n^2\Delta}\left(1 + \left[\frac{2}{3b}p^2 + \frac{1}{a} - \frac{2}{3b}\right]\left(\sqrt{n}\Delta\right)^4\right)$$
$$= \frac{D + \frac{2}{\pi^2 D}\sum_{n=1}^{N}\frac{\text{Sin}^2(n\pi D)}{n^2}}{\Delta} + \frac{2}{\pi^2 D}\sum_{n=1}^{N}\text{Sin}^2(n\pi D)\left[\frac{2}{3b}p^2 + \frac{1}{a} - \frac{2}{3b}\right]\Delta^3 \qquad (24)$$

The derivative of k_r with respect to Δ is used to calculate the optimum value of Δ:

$$\frac{dk_r}{d\Delta} = -\frac{\left[D + \frac{2}{\pi^2 D}\sum_{n=1}^{N}\frac{\text{Sin}^2(n\pi D)}{n^2}\right]}{\Delta^2} + \left(\frac{2}{\pi^2 D}\sum_{n=1}^{N}\text{Sin}^2(n\pi D)\left[\frac{2}{b}p^2 + \frac{3}{a} - \frac{2}{b}\right]\right)\Delta^2 \qquad (25)$$

Setting $\frac{dk_r}{d\Delta} = 0$ gives

$$\Delta_{opt} = \sqrt[4]{\frac{D + \frac{2}{\pi^2 D}\sum_{n=1}^{N}\frac{\text{Sin}^2(n\pi D)}{n^2}}{\frac{2}{\pi^2 D}\sum_{n=1}^{N}\text{Sin}^2(n\pi D)\left[\frac{2}{b}p^2 + \frac{3}{a} - \frac{2}{b}\right]}} \qquad (26)$$

If D = 0.5, then the formula for Δ_{opt} is given by

$$\Delta_{opt} = \sqrt[4]{\frac{0.5 + \frac{4}{\pi^2}\sum_{n=1,\text{odd}}^{N}\frac{1}{n^2}}{\frac{4}{\pi^2}\left(\frac{N+1}{2}\right)\left[\frac{2}{b}p^2 + \frac{3}{a} - \frac{2}{b}\right]}} \qquad (27)$$

Also, for large N, $\sum_{n=1,\text{odd}}^{N}\frac{1}{n^2} = \sum_{k=0}^{\frac{N-1}{2}}\frac{1}{(2k+1)^2} \to \frac{\pi^2}{8}$. So with a = 7.5 and b = 6, Δ_{opt} for this case can be re-written as

$$\Delta_{opt} = \frac{1}{\sqrt[4]{\left(\frac{N+1}{2}\right)\left(0.135p^2 + 0.027\right)}} \qquad (28)$$

Example: Push-Pull Converter

Take D = 0.5, p = 6, t_r = 2.5%, and f = 50 kHz.

$$N = \frac{35}{t_r\%} = \frac{35}{2.5} = 14$$

Choose N = 13, since N is odd.

$$\Delta_{opt} = \frac{1}{\sqrt[4]{\left(\frac{13+1}{2}\right)\left(0.135 \times 6^2 + 0.027\right)}} = 0.41$$

$$\delta_o = \frac{66}{\sqrt{f}} = \frac{66}{\sqrt{50 \times 10^3}} = 0.295 \text{ mm}$$

The optimum foil thickness is equal to

$$d_{opt} = \Delta_{opt}\delta_o = 0.41 \times 0.295 = 0.12 \text{ mm}$$

The value of k_r is calculated using the following formula derived from (24):

$$k_r = \frac{0.5 + \frac{4}{\pi^2}\sum_{n=1,\,odd}^{N}\frac{1}{n^2}}{\Delta} + \frac{4}{\pi^2}\left(\frac{N+1}{2}\right)\left(0.111p^2 + 0.022\right)\Delta^3$$

$$= \frac{0.5 + \frac{4}{\pi^2}(1.198)}{0.41} + \frac{4}{\pi^2}\left(\frac{14}{2}\right)\left(0.111(6)^2 + 0.022\right)(0.41)^3$$

$$= 3.19 \quad \ldots \quad \text{exact is } 3.12$$

The ac resistance can now be evaluated as

$$R_{eff} = k_r \Delta R_{dc} = 3.19 \times 0.41 R_{dc}$$
$$= 1.31 R_{dc} \quad \ldots \quad \text{exact is } 1.34 R_{dc}$$

CONCLUSIONS

The derivation of an approximate formula for the optimum thickness of a high frequency transformer winding has been described for the case of a rectified square waveform, and similar formulæ have been derived for other waveforms as shown in Table 1.

ACKNOWLEDGEMENTS

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REFERENCES

[1] Dowell, P.L.: "Effects of Eddy Currents in Transformer Windings", IEEE Proceedings, Vol. 113 No. 8, 1966.

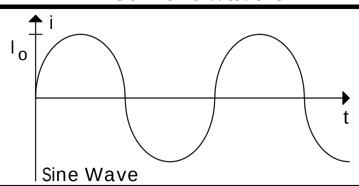
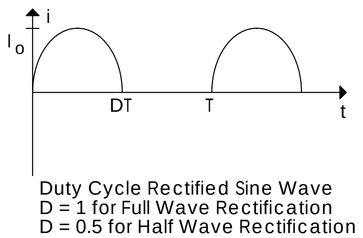
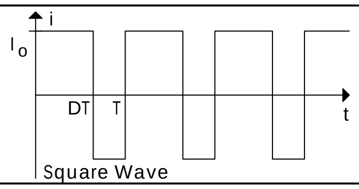
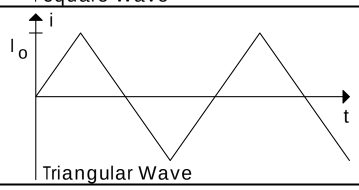
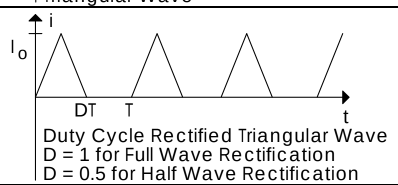
Current Waveform	Approximation Formula for Δ_{opt}
	$\Delta_{opt} = \sqrt[4]{\frac{1}{\left[\frac{2}{b}p^2 + \frac{3}{a} - \frac{2}{b}\right]}}$
	For 1/(2D) = k $\notin$ **N**, $\Delta_{opt} = \sqrt[4]{\frac{1 + \sum_{n=1}^{N}\frac{2Cos^2(n\pi D)}{\left(1 - 4n^2D^2\right)^2}}{\sum_{n=1}^{N}\frac{2Cos^2(n\pi D)}{\left(1 - 4n^2D^2\right)^2}n^2\left(\frac{2}{b}p^2 + \frac{3}{a} - \frac{2}{b}\right)}}$ For 1/(2D) = k $\in$ **N**, $\Delta_{opt} = \sqrt[4]{\frac{\frac{8D}{\pi^2} + \frac{16D}{\pi^2}\sum_{n=1,\,n\neq k}^{N}\frac{Cos^2(n\pi D)}{\left(1 - 4n^2D^2\right)^2} + \frac{1}{4k^2D}}{\left(\frac{2}{b}p^2 + \frac{3}{a} - \frac{2}{b}\right)\left[\frac{16D}{\pi^2}\sum_{n=1,\,n\neq k}^{N}\frac{Cos^2(n\pi D)}{\left(1 - 4n^2D^2\right)^2}n^2 + \frac{1}{4D}\right]}}$
	$\Delta_{opt} = \sqrt[4]{\frac{(2D-1)^2 + \frac{8}{\pi^2}\sum_{1}^{N}\frac{Sin^2(n\pi D)}{n^2}}{\frac{8}{\pi^2}\sum_{1}^{N}Sin^2(n\pi D)\left[\frac{2}{b}p^2 + \frac{3}{a} - \frac{2}{b}\right]}}$
	$\Delta_{opt} = \sqrt[4]{\frac{\sum_{n=1,\,odd}^{N}\frac{1}{n^4}}{\sum_{n=1,\,odd}^{N}\frac{1}{n^2}\left[\frac{2}{b}p^2 + \frac{3}{a} - \frac{2}{b}\right]}}$
	$\Delta_{opt} = \sqrt[4]{\frac{1 + \sum_{n=1}^{N}\frac{32}{\pi^4 n^4 D^4}Sin^4\left(\frac{n\pi D}{2}\right)}{\sum_{n=1}^{N}\frac{32}{\pi^4 n^2 D^4}Sin^4\left(\frac{n\pi D}{2}\right)\left[\frac{2}{b}p^2 + \frac{3}{a} - \frac{2}{b}\right]}}$

Table 1: Formulæ for the optimum thickness of a winding for various waveforms, a = 7.5, b = 6